

On the astrophysical scales set by the Cosmological Constant

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Abstract

It is already known that a positive Cosmological Constant Λ sets the scale $r_0 = (\frac{3}{2}r_s r_\Lambda^2)^{1/3}$, which depending on the mass of the source, can be of astrophysical order of magnitude. This scale was interpreted before as the maximum distance in order to get bound orbits. In this paper I compute r_0 with a different method and obtain its first order correction due to the angular momentum L of the test particle moving around the source. I then re derive by using more rigorous methods the maximum angular momentum in order to get bound orbits $L_{max} = \frac{1}{4}(9r_s^2 r_\Lambda)^{1/3}$ and its corresponding saddle point position given by $r_x = \frac{1}{2}(3r_s r_\Lambda^2)^{1/3}$. Here $r_s = 2GM$ is the Schwarzschild radius, $r_\Lambda = \frac{1}{\sqrt{\Lambda}}$ is the Cosmological Constant scale and β is a dimensionless parameter given by $\beta \equiv \frac{L_{max}}{L}$.

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I. INTRODUCTION

The Schwarzschild de-Sitter (S-dS) space in static coordinates has been widely studied in the past. Its analytic extension has been performed by Bażański and Ferrari [1]. They interpreted the scale $r_0 = \left(\frac{3}{2}r_s r_\Lambda\right)^{1/3}$ as the distance where the 0-0 component of the S-dS metric takes a minimum value. As a consequence of this, it was found in [2] that r_0 represents a transition distance after which a photon suffers a gravitational blue shift when it moves away from a source. The same scale is used by Bousso and Hawking in order to find the appropriate expression for the temperature of a black hole immersed inside a de-Sitter space [3]. In such a case, the distance r_0 is interpreted as the position of the static observer in order to find the appropriate normalization for the time-like Killing vector. As a consequence of this, there exist a minimum temperature for the black hole given by $T = \frac{1}{2\pi r_\Lambda}$ [3, 4]. In [5], Balaguera et al, found that r_0 represents the maximum distance within which we can find bound orbits solutions for a test particle moving around a source. In the same manuscript, the velocity bounds for a test particle inside the S-dS space were obtained, this work was then extended by Arraut et al in [6] in order to incorporate other metric solutions. In [5], the authors also found that there exist a maximum angular momentum L_{max} for the test particle to be inside a bound orbit. If $L = L_{max}$, then there exist a saddle point for the effective potential at the distance r_x . In this paper I improve the computations given in [5] and then I find the first order correction for the distance r_0 . This correction is due to the angular momentum L of the test particle moving around the source. Additionally I prove that the saddle point position r_x and as a consequence the maximum angular momentum L_{max} mathematically represents an extremal condition for the discriminant of a 4th order polynomial. In fact, the condition $L = L_{max}$ is just equivalent to say that the discriminant for the fourth order polynomial is zero $D = 0$. The paper is organized as follows: In section II, I make a review of the Schwarzschild de-Sitter metric in static coordinates, I then obtain the effective potential (U_{eff}) and analyze the validity of the coordinate system used for the present analysis. In section III I obtain the critical points of the effective potential ($\frac{dU_{eff}}{dr} = 0$), then I obtain the scale r_0 with its first order correction due to the angular momentum L in terms of a dimensionless parameter $\beta = \frac{L_{max}}{L}$. Additionally, I re derive the scales L_{max} and r_x . In section IV I compute by using the same methods of section III the zero effective potential conditions $U_{eff}(r) = 0$. The objective is to verify the consistency of the critical point results $\frac{dU_{eff}}{dr} = 0$. And finally in section V, I conclude.

II. THE SCHWARZSCHILD DE-SITTER METRIC IN STATIC COORDINATES

Schwarzschild-de Sitter metric in static coordinates, is given by:

$$ds^2 = -e^{\nu(r)} dt^2 + e^{-\nu(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (1)$$

where:

$$e^{\nu(r)} = 1 - \frac{r_s}{r} - \frac{r^2}{3r_\Lambda^2} \quad (2)$$

if we want to find equations of motion for the S-dS metric, the work is simplified if we use its symmetry. From symmetry arguments, we know that there are 4 killing vectors, three

for spatial symmetry and one for time translations. Each of these will lead to a constant of the motion for a free particle [7]. If K^μ is a Killing vector, we know that:

$$K_\mu \frac{dx^\mu}{d\lambda} = \text{Constant} \quad (3)$$

is a constant of motion. In addition, there is another constant of motion, namely, the geodesic equation (together with metric compatibility). Then the quantity:

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -\epsilon \quad (4)$$

is a constant of motion along the path [7]. In eq. 4 $\epsilon = 1$ for massive particles and $\epsilon = 0$ for massless particles. The quantity associated with invariance under spatial rotations is the angular momentum. We can think about the angular momentum as a three-vector with magnitude (one component) and direction (two components). Conservation of the direction of angular momentum means that the particle will move over a plane, we can choose this to be the equatorial plane of our coordinate system. If the particle is not initially over the plane, we can rotate the coordinate system until that condition is satisfied. Then we can choose the angle [7]:

$$\theta = \frac{\pi}{2} \quad (5)$$

we then have two remaining Killing vectors corresponding to the conserved quantity related to time translations and the magnitude of angular momentum. The quantity related with time arises from the time-like Killing vector:

$$K^\mu = (\partial_t)^\mu = (1, 0, 0, 0) \quad (6)$$

the Killing vector whose conserved quantity is the magnitude of the angular momentum is:

$$R^\mu = (\partial_\phi)^\mu = (0, 0, 0, 1) \quad (7)$$

in both cases, it is convenient to lower the index and then obtain:

$$K_\mu = (-e^{\nu(r)}, 0, 0, 0) \quad (8)$$

thus:

$$K_\mu = \left(- \left(1 - \frac{r_s}{r} - \frac{r^2}{3r_A^2} \right), 0, 0, 0 \right) \quad (9)$$

and:

$$R_\mu = (0, 0, 0, r^2) \quad (10)$$

from eq. 3, the two conserved quantities are:

$$E = -K_\mu \frac{dx^\mu}{d\lambda} = e^{\nu(r)} \frac{dt}{d\tau} \quad (11)$$

and:

$$L = R_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\tau} \quad (12)$$

where L is the magnitude of the angular momentum, and $\tau = \lambda$ is the proper time. Developing explicitly eq. 4 for massive test particles, we obtain:

$$-1 = g_{00} \left(\frac{dt}{d\tau} \right)^2 + g_{ii} \left(\frac{dx^i}{d\tau} \right)^2 \quad (13)$$

if we introduce the S-dS metric given by eq. 1, then:

$$-1 = - \left(1 - \frac{r_s}{r} - \frac{r^2}{3r_\Lambda^2} \right) \left(\frac{dt}{d\tau} \right)^2 + \left(1 - \frac{r_s}{r} - \frac{r^2}{3r_\Lambda^2} \right)^{-1} \left(\frac{dr}{d\tau} \right)^2 + r^2 \left(\frac{d\phi}{d\tau} \right)^2 \quad (14)$$

this equation is just equivalent to:

$$- \left(1 - \frac{r_s}{r} - \frac{r^2}{3r_\Lambda^2} \right) = - \left(1 - \frac{r_s}{r} - \frac{r^2}{3r_\Lambda^2} \right)^2 \left(\frac{dt}{d\tau} \right)^2 + \left(\frac{dr}{d\tau} \right)^2 + r^2 \left(1 - \frac{r_s}{r} - \frac{r^2}{3r_\Lambda^2} \right) \left(\frac{d\phi}{d\tau} \right)^2 \quad (15)$$

using eqns. 2, 11 and 12, we obtain:

$$- \left(1 - \frac{r_s}{r} - \frac{r^2}{3r_\Lambda^2} \right) = -E^2 + \left(\frac{dr}{d\tau} \right)^2 + \left(1 - \frac{r_s}{r} - \frac{r^2}{3r_\Lambda^2} \right) \frac{L^2}{r^2} \quad (16)$$

dividing by two, developing and rearranging terms, we get:

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 + \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3} - \frac{r_s}{2r} - \frac{1}{6} \frac{r^2}{r_\Lambda^2} = \frac{1}{2} \left(E^2 + \frac{L^2}{3r_\Lambda^2} - 1 \right) = C \quad (17)$$

where C is a constant depending on the initial conditions of motion, we can define the effective potential as:

$$U_{eff}(r) = -\frac{r_s}{2r} - \frac{1}{6} \frac{r^2}{r_\Lambda^2} + \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3} \quad (18)$$

then eq. 17 is equivalent to:

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 + U_{eff}(r) = \frac{1}{2} \left(E^2 + \frac{L^2}{3r_\Lambda^2} - 1 \right) = C \quad (19)$$

the first term on the right-hand side of eq. 18 is the Newtonian gravitational potential, the second term is the Λ contribution (it reproduces a repulsive effect), the third term is the centrifugal force and it takes the same form in Newtonian gravity and General Relativity. The last term is the General Relativity correction to the effective potential. This last contribution is important at short scales i.e comparable to the gravitational radius $2GM$. In this manuscript, we assume that $r_\Lambda \gg r_s$. We can use a relation between L and E , obtained as follows:

$$L = r^2 \frac{d\phi}{d\tau} = r^2 \frac{d\phi}{dt} \frac{dt}{d\tau} = r^2 \dot{\phi} e^{-\nu(r)} E \quad (20)$$

where $\dot{\phi} = \frac{d\phi}{dt}$ is the angular velocity of the test particle moving around the source. Replacing eq. 20 in eq. 17, we obtain:

$$\frac{1}{2}E^2 \left(1 + \frac{r^4 \dot{\phi}^2 e^{-2\nu(r)}}{3r_\Lambda^2} \right) = C + \frac{1}{2} \quad (21)$$

then the condition $C > -\frac{1}{2}$ must be satisfied in agreement with [5] where the same analysis was performed. Additionally, eq. 17 can be written as:

$$\frac{1}{2} \left(\frac{dr}{dt} \frac{dt}{d\tau} \right)^2 + \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3} - \frac{r_s}{2r} - \frac{1}{6} \frac{r^2}{r_\Lambda^2} = \frac{1}{2} \left(E^2 + \frac{L^2}{3r_\Lambda^2} - 1 \right) = C \quad (22)$$

using again eq. 11 and eq. 20, after a rearrangement of terms, we obtain:

$$E^2 = \frac{e^{3\nu(r)}}{\left(e^{2\nu(r)} - 2K + \dot{\phi}^2 r^2 (1 - e^{\nu(r)}) \right)} \quad (23)$$

where we have defined $2K = \dot{r}^2 + r^2 \dot{\phi}^2$ as twice the Kinetic energy value. This equation offers certain restrictions, since we need E^2 to be positive definite. The condition $E=0$ is equivalent to the condition $e^{\nu(r)} = 0$, namely, the event horizon condition. Then the expression 23 shows the regime of validity of our coordinate system. The event horizons in S-dS metric, are obtained under the condition:

$$0 = 1 - \frac{r_s}{2r} - \frac{r^2}{3r_\Lambda^2} \quad (24)$$

it is already known that the solution of eq. 24 provides two real solutions given by:

$$r_{CH} = -2r_\Lambda \cos \left(\frac{1}{3} \left(\cos^{-1} \left(\frac{3r_s}{2r_\Lambda} \right) + 2\pi \right) \right) \quad (25)$$

and:

$$r_{BH} = -2r_\Lambda \cos \left(\frac{1}{3} \left(\cos^{-1} \left(\frac{3r_s}{2r_\Lambda} \right) + 4\pi \right) \right)$$

where r_{CH} is the Cosmological Horizon and r_{BH} is the Black Hole event horizon. In this paper I will work in the region between the two horizons. I also assume $r_\Lambda \gg r_s$. If $r_\Lambda \sim r_s$, then $r_{CH} \sim r_{BH}$ and then our coordinate system is inappropriate [3]. I will not consider such situations in this manuscript. Under the condition $r_\Lambda \gg r_s$, eqns. 25 become [2]:

$$r_{CH} \approx \sqrt{3}r_\Lambda - \frac{1}{2}r_s \quad (26)$$

and:

$$r_{BH} \approx r_s + \frac{1}{6} \frac{r_s^4}{r_\Lambda^3}$$

for completeness, the angular momentum in terms of the variables introduced in eq. 23 is:

$$L^2 = \frac{r^4 \dot{\phi}^2 e^{3\nu(r)}}{\left(e^{2\nu(r)} - 2K + \dot{\phi}^2 r^2 (1 - e^{\nu(r)})\right)} \quad (27)$$

as can be easily verified. Additionally, the constant C , can be expressed as:

$$C(r_i, v_i, \dot{\phi}_i) = \frac{v_i^2 + 2(e^{2\nu(r_i)} + v_{\phi i}^2) U_{eff}(r_i)_{L=0} + \frac{r_i^2 v_{\phi i}^2 e^{\nu(r_i)}}{3r_A^2}}{2(e^{2\nu(r_i)} - v_i^2 - 2v_{\phi i}^2 U_{eff}(r_i)_{L=0})} \quad (28)$$

where $U_{eff}(r_i)_{L=0} = -\frac{r_s}{2r_i} - \frac{1}{6} \frac{r_i^2}{r_A^2}$, and the subindex i denotes initial conditions. v is the total velocity of a test particle, and $v_\phi = r\dot{\phi}$ is its tangential velocity. Eq. 28 as $L \rightarrow 0$, recovers the result obtained in ref. [5], where the same analysis for the case $L = 0$ was performed.

III. CIRCULAR ORBIT CONDITIONS FOR THE EFFECTIVE POTENTIAL WITH Λ

We can now obtain the critical points for the effective potential obtained from eq. 18. I will start the analysis with the case where the Cosmological Constant goes to zero ($\Lambda = 0$). We know that the effective potential for a massive test particle is given by [7]:

$$U_{eff}(r)_{\Lambda=0} = -\frac{r_s}{2r} + \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3} \quad (29)$$

taking the derivative with respect to r of this equation we then obtain:

$$\frac{dU_{eff}(r)_{\Lambda=0}}{dr} = \frac{r_s}{2r^2} - \frac{L^2}{r^3} + \frac{3r_s L^2}{2r^4} \quad (30)$$

in order to find the circular orbits, we have to send to zero eq. 30, then:

$$r_c^2 - \frac{L^2}{r_s} r_c + 6L^2 = 0 \quad (31)$$

where we have defined r_c as the distance at which the orbits for a test particle moving around the source are circular. The solutions of this previous equation are:

$$r_c = \frac{L^2 \pm \sqrt{L^4 - 3r_s^2 L^2}}{r_s} \quad (32)$$

if $L \rightarrow \infty$, then:

$$r_c \approx \frac{L^2 \pm L^2 \left(1 - \frac{3r_s^2}{2L^2}\right)}{r_s} = \left(\frac{2L^2}{r_s}, \frac{3}{2}r_s\right) \quad (33)$$

the first root is the stable orbit [7] and the second root is proportional to the Schwarzschild radius and it is the unstable equilibrium orbit. The minimum angular momentum in order to get bound orbits is obtained as the discriminant of eq. 32 is zero, in such a case:

$$L_{min} = \sqrt{3}r_s \quad (34)$$

when the angular momentum takes this value, the two distances for circular orbits obtained in eq. 32 become the same and they correspond to the saddle point position given by:

$$r_{cx} = 3r_s \quad (35)$$

the same result is obtained if the first and second derivative of the effective potential given by eq. 29 go to zero (See appendix A). If we replace the second root of given in eq. 33 ($r_c = \frac{3}{2}r_s$) inside eq. 30, then we obtain:

$$\frac{dU_{eff}(r)}{dr} \Big|_{\Lambda=0} = \frac{2}{9r_s} - \frac{8L^2}{27r_s^3} + \frac{8L^2}{27r_s^3} = \frac{2}{9r_s} \quad (36)$$

we then conclude that $r_c = \frac{3}{2}r_s$ is not an exact solution for eq. 31. This distance becomes to be an exact solution only under the approximation $L \gg r_s$. In such a case, then the term $\frac{r_s}{r^2}$ must be ignored from eq. 30 at short distances r . By the same argument, the term $-\frac{r_s}{r}$ must be dropped from eq. 29. Then for short scales ($r \rightarrow r_s$), the effective potential as $\Lambda = 0$ can be approximated:

$$U_{eff}(r)_{\Lambda=0} \approx \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3} \quad (37)$$

ignoring then the standard Newtonian contribution. As a proof of this statement, we can take the derivative with respect to r from the previous equation:

$$\frac{dU_{eff}(r)}{dr} \Big|_{\Lambda=0} \approx -\frac{L^2}{r^3} + \frac{3r_s L^2}{2r^4} \quad (38)$$

if we set this result to zero, then we obtain:

$$r_1 = \frac{3}{2}r_s \quad (39)$$

which is just the second root given in eq. 33. Fig. 1 shows the effective potential curve for distance scales around the gravitational radius GM . If we accept the fact that the term $-\frac{r_s}{r}$ is negligible for short distances r in the equation 29 as far as the approximation $L \gg r_s$ is valid, then the same argument will be valid as $\Lambda \neq 0$. And the additional contribution to the effective potential given by $-\frac{1}{6}\frac{r^2}{r_\Lambda^2}$, will be also negligible for short distances r . Later we will see that the Λ effects become important for distances of the order of magnitude of $r_0 = \left(\frac{3}{2}r_s r_\Lambda^2\right)^{1/3}$ or larger. The full effective potential energy with the inclusion of the Λ term is given by:

$$U_{eff}(r) = -\frac{r_s}{2r} - \frac{r^2}{6r_\Lambda^2} + \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3} \quad (40)$$

under the previous approximations, the first circular orbit which is the unstable one will be located at the same distance $r_1 = \frac{3}{2}r_s$, even for the case $\Lambda \neq 0$ as far as the condition $r_\Lambda \gg L \gg r_s$ is satisfied. Figure 1 shows the effective potential for distance scales of the order of magnitude of the gravitational radius r_s . If we consider the second position for circular orbits given by $r_2 = \frac{2L^2}{r_s}$ (see eq. 33) as $\Lambda = 0$, it is then possible to verify that it is not an exact solution of eq. 30 neither. In fact, by replacing r_2 in the mentioned equation, we then obtain:

$$\frac{dU_{eff}(r)_{\Lambda=0}}{dr} = \frac{3r_s^5}{32L^6} \quad (41)$$

then, r_2 is not an exact root for the case $\Lambda = 0$. It is easy to demonstrate that r_2 becomes an exact solution only if the term $\frac{3r_s L^2}{2r^4}$ (GR contribution) from eq. 30 is neglected. Then it is necessary to ignore the term $-\frac{r_s L^2}{2r^3}$ in eq. 29 in such a case. Then we conclude that for large distances r , as $\Lambda = 0$, under the assumption $L \gg r_s$, the effective potential ($\Lambda = 0$) can be approximated as:

$$U_{eff}(r)_{\Lambda=0} \approx -\frac{r_s}{2r} + \frac{L^2}{2r^2} \quad (42)$$

if we take the derivative of this expression with respect to r and send it to zero, we get:

$$\frac{dU_{eff}(r)}{dr} \Big|_{\Lambda=0} \approx \frac{r_s}{2r^2} - \frac{L^2}{r^3} = 0 \quad (43)$$

then the solution for this equation is of course r_2 as it should be. This result will be important at the moment of making the computations for the case $\Lambda \neq 0$. From eq. 40, we see that the Λ term should be important for large values of r . If we still satisfy the condition $L \gg r_s$, then the effective potential for the case $\Lambda \neq 0$ and for large scales can be approximated like:

$$U_{eff}(r) \approx -\frac{r_s}{2r} - \frac{r^2}{6r_A^2} + \frac{L^2}{2r^2} \quad (44)$$

where we have neglected the general Relativity correction term $-\frac{r_s L^2}{2r^3}$ in agreement with the previous assumptions. If we take the derivative with respect to r of the previous equation, we obtain:

$$\frac{dU_{eff}(r)}{dr} \approx \frac{r_s}{2r^2} - \frac{r}{3r_A^2} - \frac{L^2}{r^3} = 0 \quad (45)$$

multiplying this equation with r^3 and rearranging terms, we get:

$$r^4 - \frac{3}{2}r_A^2 r_s r + 3r_A^2 L^2 = 0 \quad (46)$$

this equation is a reduced fourth order polynomial in agreement with appendix B. We can make a comparison with the standard reduced form given by equation B7 of appendix B. Here I rewrite the equation in terms of the variable r as:

$$r^4 + Pr^2 + Qr + K = 0 \quad (47)$$

if we compare this equation with the result 46, then we can define the coefficients:

$$P = 0 \quad Q = -\frac{3}{2}r_A^2 r_s \quad K = 3r_A^2 L^2 \quad (48)$$

we can extract the associated cubic equation. This is given by eq. B9, which in this case becomes:

$$z^3 - 12r_A^2 L^2 z - \frac{9}{4}r_A^4 r_s^2 = 0 \quad (49)$$

in agreement with the coefficients defined in eq. 48. The solutions for the fourth-order equation 46 in terms of the associated solutions of eq. 49, can be found from eq. B10, which here is just rewritten in terms of the variable r as:

$$\begin{aligned} r_1^* &= \frac{1}{2}(\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}) \\ r_2^* &= \frac{1}{2}(\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}) \\ r_3^* &= \frac{1}{2}(-\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}) \\ r_4^* &= \frac{1}{2}(-\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}) \end{aligned} \tag{50}$$

also the solutions of the associated third-order equation must satisfy the condition B11 from appendix B. Q has been defined in eq. 47 and in this particular case is given by the result 48. Our objective here is to find the real roots for the fourth-order equation. We can compare eq. 49, which is already in a reduced form, with the standardized reduced form given in eq. C5 from appendix C, which is rewritten here in terms of the variable z as:

$$z^3 + pz + q = 0 \tag{51}$$

where p , and q are constant coefficients. In our case, from eq. 49 we obtain:

$$p = -12r_A^2 L^2 \quad q = -\frac{9}{4}r_A^4 r_s^2 \tag{52}$$

it is necessary to find the discriminant D which is given in eq. C7, rewritten here as:

$$D \equiv \frac{1}{27}p^3 + \frac{1}{4}q^2 \tag{53}$$

where p and q are given by eq. 52, replacing, we have:

$$D = -64r_A^6 L^6 + \frac{81}{64}r_A^8 r_s^4 \tag{54}$$

the solution for the associated cubic equation given in 51, depends on the sign of the discriminant defined in eq. 54. Can be demonstrated that if $D < 0$, then there are no real solutions for the fourth-order equation 46. This case then is not important for us. However if $D > 0$, we can get real solutions. Then $D = 0$ must provide some limit condition given by:

$$0 = -64r_A^6 L^6 + \frac{81}{64}r_A^8 r_s^4 \tag{55}$$

if we solve for L , we obtain:

$$L_{max} = \frac{3^{2/3}}{4}(r_s^2 r_A)^{1/3} \tag{56}$$

this condition is coincident with a result obtained in [5]. In such a case, it was interpreted as the maximum angular momentum for a test particle to be inside a bound orbit. Here that

interpretation is verified by the fact that for the case $L > L_{max}$ there are no real solutions for the fourth order polynomial. We can now concentrate on the case $D > 0$. We also know from eq. 52 that the conditions $p < 0$ and $q < 0$ are satisfied. If we want to find the solution for the associated third-order equation 51, then we need to define first the parameter given in eq. C8 of appendix C. In this particular case that parameter R depends on the angular momentum and it is given by:

$$R = -2r_A L \quad (57)$$

here we use the case ii) of appendix C, specifically by eq. C11. From eq. 52 and eq. 57, we obtain the auxiliary angle:

$$\cosh \phi = \frac{9}{64} \frac{r_A r_s^2}{L^3} \quad (58)$$

solving for ϕ gives:

$$\phi = \cosh^{-1} \left(\frac{9}{64} \frac{r_A r_s^2}{L^3} \right) \quad (59)$$

if we introduce eq. 56 in this result, then the auxiliary angle can be expressed as:

$$\phi = \cosh^{-1} \left(\frac{L_{max}}{L} \right)^3 \quad (60)$$

the three solutions for the associated cubic equation 49, are obtained from eq. C12 of appendix C. However in this case $y = z$, so we get for the first root:

$$z_1 = -2R \cosh \left(\frac{\phi}{3} \right) \quad (61)$$

replacing the results 57 and 60, we get:

$$z_1 = 4r_A L \cosh \left(\frac{1}{3} \cosh^{-1} \left(\frac{L_{max}}{L} \right)^3 \right) \quad (62)$$

the second root of the associated cubic equation 49 is:

$$z_2 = R \cosh \left(\frac{\phi}{3} \right) + i\sqrt{3}R \sinh \left(\frac{\phi}{3} \right) \quad (63)$$

replacing again the results 57 and 60 we obtain:

$$z_2 = -2r_A L \cosh \left(\frac{1}{3} \cosh^{-1} \left(\frac{L_{max}}{L} \right)^3 \right) - 2i\sqrt{3}r_A L \sinh \left(\frac{1}{3} \cosh^{-1} \left(\frac{L_{max}}{L} \right)^3 \right) \quad (64)$$

finally, the third root is given by:

$$z_3 = z_2^* = R \cosh \left(\frac{\phi}{3} \right) - i\sqrt{3}R \sinh \left(\frac{\phi}{3} \right) \quad (65)$$

thus by analogous procedure, we get:

$$z_3 = z_2^* = -2r_\Lambda L \cosh \left(\frac{1}{3} \cosh^{-1} \left(\frac{L_{max}}{L} \right)^3 \right) + 2i\sqrt{3}r_\Lambda L \sinh \left(\frac{1}{3} \cosh^{-1} \left(\frac{L_{max}}{L} \right)^3 \right) \quad (66)$$

we can define the dimensionless parameter $L \equiv \frac{L_{max}}{\beta}$. Then eqns. 62, 64 and 66 are equivalent to:

$$z_1 = 4 \frac{r_\Lambda L_{max}}{\beta} \cosh \left(\frac{1}{3} \cosh^{-1}(\beta^3) \right) \quad (67)$$

$$z_2 = -2 \frac{r_\Lambda L_{max}}{\beta} \cosh \left(\frac{1}{3} \cosh^{-1}(\beta^3) \right) - 2i\sqrt{3} \frac{r_\Lambda L_{max}}{\beta} \sinh \left(\frac{1}{3} \cosh^{-1}(\beta^3) \right) \quad (68)$$

and:

$$z_3 = -2 \frac{r_\Lambda L_{max}}{\beta} \cosh \left(\frac{1}{3} \cosh^{-1}(\beta^3) \right) + 2i\sqrt{3} \frac{r_\Lambda L_{max}}{\beta} \sinh \left(\frac{1}{3} \cosh^{-1}(\beta^3) \right) \quad (69)$$

if $\beta = 1$, then $L = L_{max}$ and:

$$z_1 = 4r_\Lambda L_{max} \quad (70)$$

$$z_2 = -2r_\Lambda L_{max} = z_3 \quad (71)$$

additionally, the solutions must satisfy the condition B11. If we take the root square of eqns. 70 and 71. Then the condition B11 becomes:

$$2\sqrt{r_\Lambda L_{max}} \sqrt{-2r_\Lambda L_{max}} \sqrt{-2r_\Lambda L_{max}} = Q = -\frac{3}{2} r_s r_\Lambda^2 \quad (72)$$

the left hand side of this equation becomes $\sqrt{-2r_\Lambda L_{max}} = i\sqrt{2r_\Lambda L_{max}}$. Then eq. 72 is:

$$-4(r_\Lambda L_{max})^{3/2} = -\frac{3}{2} r_s r_\Lambda^2 \quad (73)$$

which is consistent with the maximum value of the angular momentum found in eq. 56. If we replace the results 70 and 71 in eq. 50, then there is only one real solution for the fourth order equation given by:

$$r_1^* = r_2^* = \sqrt{r_\Lambda L_{max}} = r_x \quad (74)$$

where r_x is the the saddle point position for astrophysical scales. If we use eq. 56 in the previous solutions, then we get:

$$r_1^* = r_2^* = r_x = \frac{(3)^{1/3}}{2} (r_s r_\Lambda^2)^{1/3} \quad (75)$$

which is the same result obtained in [5] by using different methods. We will now expand $\cosh^{-1}(\beta^3)$ by using the approximation:

$$\cosh^{-1}x = \ln \left(x + \sqrt{x^2 - 1} \right) \quad (76)$$

if $x^2 \gg 1$, then:

$$\cosh^{-1}x \approx \ln(2x) \quad (77)$$

in our particular case $x = \beta^3$. Then we can use the approximation as $\beta^6 \gg 1$:

$$\cosh^{-1}(\beta^3) = \ln(2\beta^3) \quad (78)$$

then we can study easily the cases for which the condition $\beta^6 \gg 1$ is satisfied. This condition is equivalent to say that $L_{max}^3 \gg L^3$. Then we can approximate eq. 62 like:

$$z_1 \simeq 4 \frac{r_\Lambda L_{max}}{\beta} \cosh(\ln(2)^{1/3} \beta) \quad (79)$$

thus:

$$z_1 \simeq 2r_\Lambda L_{max} \left((2)^{1/3} + \frac{1}{(2)^{1/3} \beta^2} \right) \quad (80)$$

the root square of this result is:

$$\sqrt{z_1} = \sqrt{2r_\Lambda L_{max} \left((2)^{1/3} + \frac{1}{(2)^{1/3} \beta^2} \right)} \quad (81)$$

under the condition $\beta^2 \gg 1$, an expansion around $\frac{1}{\beta^2}$ can be performed and we get:

$$\sqrt{z_1} = 4^{1/3} \sqrt{r_\Lambda L_{max}} + \frac{1}{2\beta^2} \sqrt{r_\Lambda L_{max}} \quad (82)$$

doing the same for z_2 given by eq. 64, we obtain:

$$z_2 \simeq -2 \frac{r_\Lambda L_{max}}{\beta} \cosh(\ln(2)^{1/3} \beta) - 2\sqrt{3}i \frac{r_\Lambda L_{max}}{\beta} \sinh(\ln(2)^{1/3} \beta) \quad (83)$$

then:

$$z_2 \simeq -r_\Lambda L_{max} \left((2)^{1/3} + \frac{1}{(2)^{1/3} \beta^2} \right) - \sqrt{3}ir_\Lambda L_{max} \left((2)^{1/3} - \frac{1}{(2)^{1/3} \beta^2} \right) \quad (84)$$

if we take the root square for this result, then:

$$\sqrt{z_2} \simeq i \sqrt{r_\Lambda L_{max} \left((2)^{1/3} + \frac{1}{(2)^{1/3} \beta^2} \right)} + \sqrt{3}ir_\Lambda L_{max} \left((2)^{1/3} - \frac{1}{(2)^{1/3} \beta^2} \right) \quad (85)$$

this result can be translated into a complex number in polar representation given by:

$$\sqrt{z_2} \simeq i \sqrt{2(2)^{1/3} r_\Lambda L_{max} e^{i\pi/3} + \frac{(2)^{2/3} r_\Lambda L_{max}}{\beta^2} e^{-i\pi/3}} \quad (86)$$

Expanding again in a Taylor series, we get:

$$\sqrt{z_2} \simeq i \left(4^{1/3} \sqrt{r_\Lambda L_{max}} e^{i\pi/6} + \frac{1}{2} \frac{\sqrt{r_\Lambda L_{max}}}{\beta^2} e^{-i\pi/2} \right) \quad (87)$$

by using the same procedure for z_3 :

$$\sqrt{z_3} \simeq i \left(4^{1/3} \sqrt{r_\Lambda L_{max}} e^{-i\pi/6} + \frac{1}{2} \frac{\sqrt{r_\Lambda L_{max}}}{\beta^2} e^{i\pi/2} \right) \quad (88)$$

where we used the condition $z_3 = z_2^*$. Then, using again the result 50, we get as a real solutions:

$$r_1^* = \frac{3}{4} \frac{1}{\beta^2} \sqrt{r_\Lambda L_{max}} \quad (89)$$

inserting the result 56 in this previous equation, we obtain:

$$r_1^* = \frac{2L^2}{r_s} \quad (90)$$

this result is just the standard circular stable orbit obtained even for the case $\Lambda = 0$. Fig. 2 illustrate the behavior of the effective potential for intermediate distances scales. The stable orbit condition is clearly perceived from the curve. Here we are working under the condition $\beta^2 \gg 1$ is satisfied. Additionally, there is a second real solution for the equation 46. This is r_2^* in eq. 50 given explicitly as:

$$r_2^* = 4^{1/3} \sqrt{r_\Lambda L_{max}} - \frac{1}{2\beta^2} \sqrt{r_\Lambda L_{max}} \quad (91)$$

using again eq. 56, we obtain:

$$r_2^* = \left(\frac{3}{2} r_s r_\Lambda^2 \right)^{1/3} - \frac{1}{4\beta^2} (3r_s r_\Lambda^2)^{1/3} = r_0' \quad (92)$$

this distance can be of astrophysical order of magnitude. Fig. 3 illustrates the behavior of the effective potential for distances near r_0' . In this figure it is clear the departure with respect to the behavior found in the standard Schwarzschild case. If $\beta^2 \gg 1$, then:

$$r_2^* \approx \left(\frac{3}{2} r_s r_\Lambda^2 \right)^{1/3} \quad (93)$$

consistent with the results obtained in [5] as $L \rightarrow 0$. On the other hand, it is possible to obtain an approximate relation between r_1^* and r_2^* . If we replace eq. 56 in eq. 89, then we get:

$$r_1^* = \left(\frac{3}{8\beta^2} \right) (3r_s r_\Lambda^2)^{1/3} \quad (94)$$

if we solve for β , then:

$$\beta^2 = \left(\frac{3}{8} \right) \frac{(3r_s r_\Lambda^2)^{1/3}}{r_1^*} \quad (95)$$

if we replace this in eq. 92, then:

$$r_2^* = \left(\frac{3}{2} r_s r_\Lambda^2 \right)^{1/3} - \frac{2}{3} r_1^* \quad (96)$$

this is an approximate relation among r_1^* and r_2^* which correspond to circular orbits, one stable (r_1^*) and the other unstable r_2^* and due to a local effect of Λ . This previous relation is valid under the condition $L \gg r_s$. Figure 2 shows the position for the stable circular orbit and 3 shows the same but for the unstable one.

IV. DISTANCES FOR ZERO ENERGY EFFECTIVE POTENTIAL

I will study the scales for which the effective potential given in eq. 40 is just zero. The procedure will be analogous to that performed in the previous section, taking the convenient approximation when we are at short distances or at large distances from the source. The general equation to be solved now is:

$$U_{eff}(r) = -\frac{r_s}{2r} - \frac{r^2}{6r_A^2} + \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3} = 0 \quad (97)$$

again for short distances r relative to the source, if $L \gg r_s$ then eq. 37 is valid and the previous condition reduces to:

$$U_{eff}(r) \approx \frac{L^2}{2r^2} - \frac{r_s L^2}{2r^3} = 0 \quad (98)$$

this equation is easy to solve, the result is simply:

$$r_1^Z = r_s \quad (99)$$

where the super index means Zero. Figure 1 shows the plots of the effective potential for short distances in comparison with r_0 . Different plots correspond to different values for the angular momentum.

On the other hand, for large distances relative to the source, we again use eq. 44 and then obtain:

$$U_{eff}(r) \approx -\frac{r_s}{2r} - \frac{r^2}{6r_A^2} + \frac{L^2}{2r^2} = 0 \quad (100)$$

this equation can be transformed as:

$$r^4 + 3r_A^2 r_s r - 3r_A^2 L^2 = 0 \quad (101)$$

working with the same procedure used in the last section, from equation 47 of section III, we get:

$$P = 0 \quad ; \quad Q = 3r_A^2 r_s \quad ; \quad K = -3r_A^2 L^2 \quad (102)$$

from eq. B9 in appendix B, we have:

$$z^3 + 12r_A^2 L^2 z - 9r_A^4 r_s^2 = 0 \quad (103)$$

comparing this equation with eq. 51, we have:

$$p = 12r_A^2 L^2 \quad ; \quad q = -9r_A^4 r_s^2 \quad (104)$$

then from equation 53, we have:

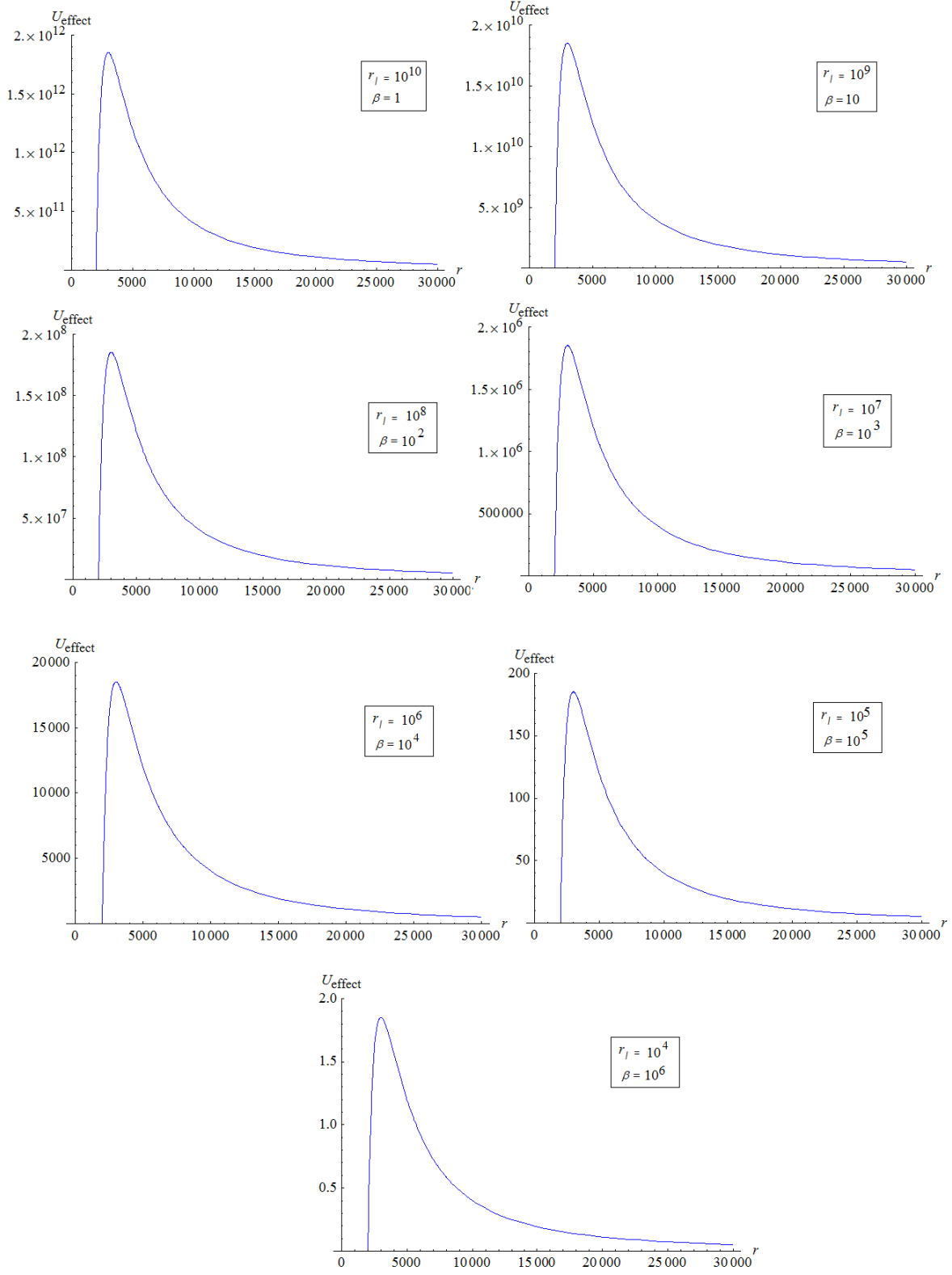


FIG. 1: Effective potential for distances of the order of the gravitational radius r_s and $L \gg r_s$.

$$D = 64r_A^6 L^6 + \frac{81}{4} r_A^8 r_s^4 > 0 \quad (105)$$

from eq. C8, we obtain:

$$R = -\sqrt{4r_A^2 L^2} = -2r_A L \quad (106)$$

for $D > 0$ and $p > 0$ which is the case here. From appendix C, we need to define the auxiliary angle to be:

$$\sinh\phi \equiv \frac{q}{2R^3} \quad (107)$$

if we replace eq. 104 and eq. 106 in this previous equation, then:

$$\sinh\phi = \frac{9}{16} \frac{r_A r_s^2}{L^3} \quad (108)$$

from eq. 56, we already know:

$$L_{max}^3 = \frac{9}{64} (r_s^2 r_A) \quad (109)$$

it is convenient to write this equation as:

$$4L_{max}^3 = \frac{9}{16} r_s^2 r_A \quad (110)$$

replacing this in eq. 108:

$$\sinh\phi = 4 \left(\frac{L_{max}}{L} \right)^3 \quad (111)$$

if we use the definition $\beta \equiv \frac{L_{max}}{L}$ then:

$$\sinh\phi = 4\beta^3 \quad (112)$$

and then solving for β , we get:

$$\phi = \sinh^{-1}(4\beta^3) \quad (113)$$

using now equations C14 with $z = y$, and introducing eq. 106 and eq. 113 we obtain the solutions for the associated cubic equation:

$$z_1 = 4r_A \frac{L_{max}}{\beta} \sinh \left(\frac{1}{3} \sinh^{-1}(4\beta^3) \right) \quad (114)$$

$$z_2 = -2r_A \frac{L_{max}}{\beta} \sinh \left(\frac{1}{3} \sinh^{-1}(4\beta^3) \right) - 2\sqrt{3}ir_A \frac{L_{max}}{\beta} \cosh \left(\frac{1}{3} \sinh^{-1}(4\beta^3) \right) \quad (115)$$

and:

$$z_3 = -2r_A \frac{L_{max}}{\beta} \sinh \left(\frac{1}{3} \sinh^{-1}(4\beta^3) \right) + 2\sqrt{3}ir_A \frac{L_{max}}{\beta} \cosh \left(\frac{1}{3} \sinh^{-1}(4\beta^3) \right) \quad (116)$$

following the procedures of section III, I will analyze two cases:

Case i). $\beta = 1$

For this case, equation 114 becomes:

$$z_1 = 4r_\Lambda L_{max} \sinh \left(\frac{1}{3} \sinh^{-1}(4) \right) \quad (117)$$

thus:

$$z_1 \approx 3r_\Lambda L_{max} \quad (118)$$

the same procedure on eq. 115, and we obtain:

$$z_2 = -2r_\Lambda L_{max} \sinh \left(\frac{1}{3} \sinh^{-1}(4) \right) - 2\sqrt{3}ir_\Lambda L_{max} \cosh \left(\frac{1}{3} \sinh^{-1}(4) \right) \quad (119)$$

which is approximately:

$$z_2 \approx -\frac{3}{2}r_\Lambda L_{max} - \frac{5}{2}\sqrt{3}ir_\Lambda L_{max} \quad (120)$$

or in polar coordinates:

$$z_2 \approx \sqrt{21}e^{i(7/5)\pi}r_\Lambda L_{max} \approx \sqrt{21}e^{-i(6/10)\pi}r_\Lambda L_{max} \quad (121)$$

eq. 116 then becomes:

$$z_3 \approx -\frac{3}{2}r_\Lambda L_{max} + \frac{5}{2}\sqrt{3}ir_\Lambda L_{max} = z_2^* \quad (122)$$

or in polar coordinates:

$$z_3 \approx \sqrt{21}e^{-i(7/5)\pi}r_\Lambda L_{max} \approx \sqrt{21}e^{i(6/10)\pi}r_\Lambda L_{max} \quad (123)$$

the three solutions must satisfy the condition B11. In fact, by replacing eqns. 118, 121 and 123 inside the mentioned condition, we get:

$$\sqrt{3r_\Lambda L_{max}}2(21^{1/4})\sqrt{r_\Lambda L_{max}}\sqrt{r_\Lambda L_{max}} \approx 3r_s r_\Lambda^2 \quad (124)$$

after replacement of eq. 56, this result is in agreement with eq. 102. Then the roots corresponding to the fourth-order equation 101 are given in this case by eqns. 50, but only the last of these group of equations gives us a real and non-negative solution as can be easily checked. If we replace eqns. 118, 121 and 123 inside eq. 50, then:

$$r_4^Z = \frac{1}{2} \left(-\sqrt{3r_\Lambda L_{max}} - (21)^{1/4} \sqrt{r_\Lambda L_{max}} e^{i(7/10)\pi} - (21)^{1/4} e^{-i(7/10)\pi} \right) \quad (125)$$

by using the identity $e^{i\theta} = \cos\theta + i\sin\theta$, then:

$$r_4^Z = \frac{1}{2} \left((-\sqrt{3r_\Lambda L_{max}} - 2(21)^{1/4} \sqrt{r_\Lambda L_{max}} \cos \left(\frac{7\pi}{10} \right)) \right) \quad (126)$$

or:

$$r_4^Z = 0, 4\sqrt{r_\Lambda L_{max}} \approx 4 \times 10^{-1} \sqrt{r_\Lambda L_{max}} \quad (127)$$

from eqns 74 and 75, the previous equation can be written like:

$$r_4^Z = 4 \times 10^{-1} r_x \quad (128)$$

where r_x symbolizes the saddle point position previously defined. Then the effective potential goes to zero before the saddle point position as it must be. Then at the saddle point position, the effective potential is negative. In order to verify it, we can evaluate the effective potential 40 at the saddle point position 75 and we then obtain:

$$U_{eff}(r_x) = -\frac{r_s}{2r_x} - \frac{1}{6} \frac{r_x^2}{r_\Lambda^2} + \frac{L^2}{2r_x^2} - \frac{r_s L^2}{2r_x^3} \quad (129)$$

but at the saddle point, the distances are large enough and we can neglect the GR correction term as far as the condition $L \gg r_s$ is satisfied. Then if we consider eq. 44, after replacing the result 75 then:

$$U_{eff}(r_x) \approx -\frac{r_s}{2r_x} - \frac{r_x^2}{6r_\Lambda^2} + \frac{L^2}{2r_x^2} \approx -\left(\frac{r_s}{r_\Lambda}\right)^{2/3} < 0 \quad (130)$$

then the effective potential at the saddle point position is negative under the previous assumptions. Although it is in general very small in magnitude since the term $\frac{r_s}{r_\Lambda} \ll 1$, in fact this term is of order 10^{-15} as $M \sim M_{sun}$ (the solar mass), however its value depends of the mass of the source producing the gravitational attraction. Eq. 128 can be expressed in terms of the maximum angular momentum given in eq. 56, then:

$$r_2^Z \approx \frac{L_{max}^2}{r_s} \approx \frac{L_{max}^2}{r_s} \quad (131)$$

Fig. 2, shows the zero for the effective potential located at scales of the order of magnitude of r_0 . From Fig. 3, it is clear that there are no zeros for distances larger then r_0 . For $\beta = 1$ we know at what positions the effective potential goes to zero (see eqns. 99 and 128), we know the position for the saddle point (see eq. 75). We know at what position the effective potential takes a maximum value (see eq. 39) and we we also know what is the value for the effective potential at the saddle point (see eq. 130). For a final picture of the important points for the effective potential curve, we need to find the value for the effective potential when it is evaluated at the distance given by eq. 39. This distance corresponds to a local maximum and is located at short distances r with respect to the source. Then we need to use the approximation given by eq. 37, which evaluated at the distance 39 gives:

$$U_{eff}\left(\frac{3}{2}r_s\right) \approx \frac{2}{27} \left(\frac{L}{r_s}\right)^2 \quad (132)$$

at the maximum angular momentum value 56, we get:

$$U_{eff}\left(\frac{3}{2}r_s\right) \approx \frac{3^{4/3}}{216} \left(\frac{r_\Lambda}{r_s}\right)^{2/3} > 0 \quad (133)$$

which is in general much bigger than one.

Case ii). $\beta > 1$

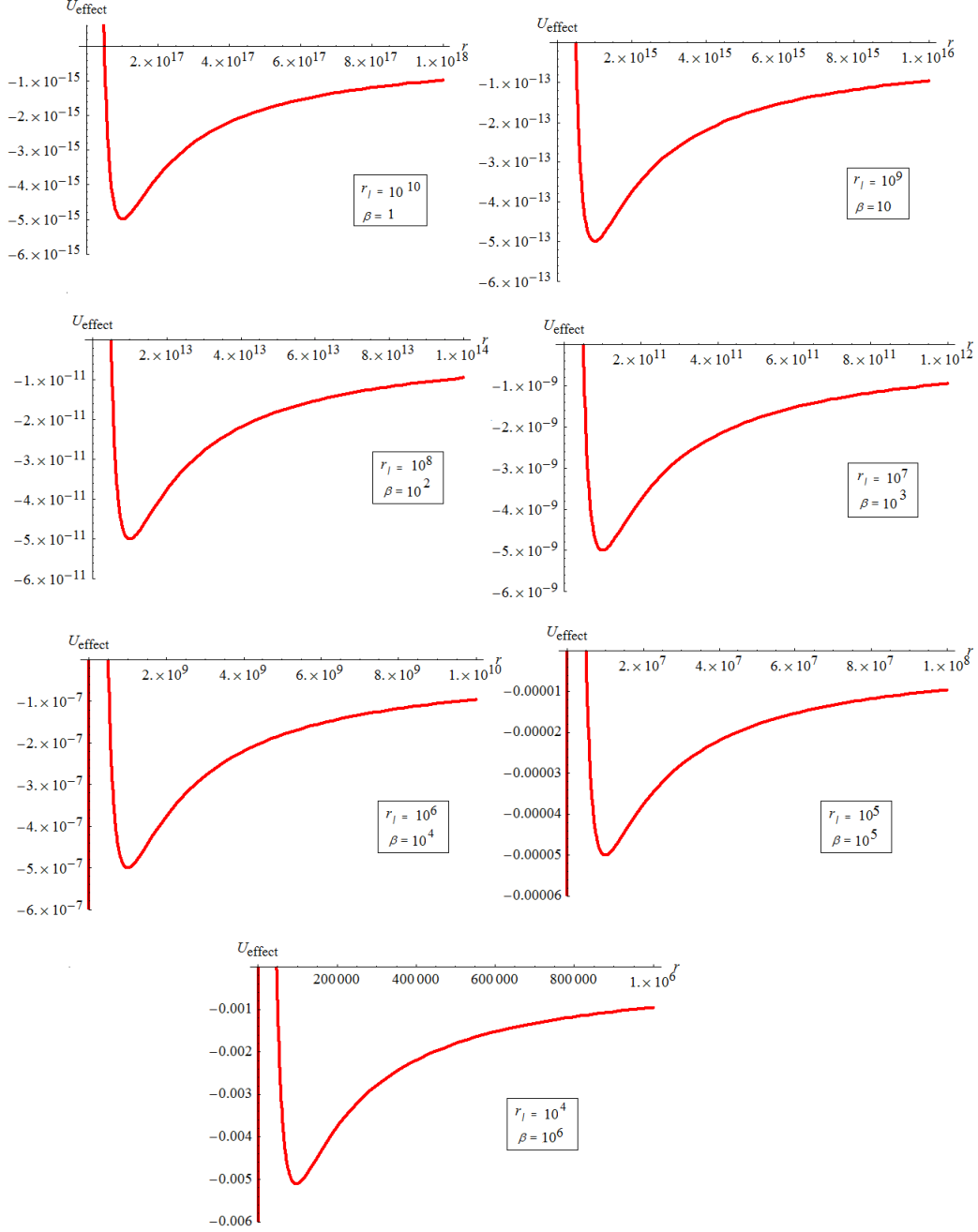


FIG. 2: Effective potential for scales $r_s \ll r \ll r_0$ and $L \gg r_s$.

Let's study now the general case for any $\beta \gg 1$. Then we can use the approximation $\sinh^{-1}(4\beta^3) \approx \ln(8\beta^3)$. With this in mind, inserting this approximation in eqns. 114, 115 and 116, we obtain:

$$z_1 \approx 4 \frac{r_\Lambda L_{max}}{\beta} \sinh(\ln(8\beta^3)^{1/3}) \quad (134)$$

thus:

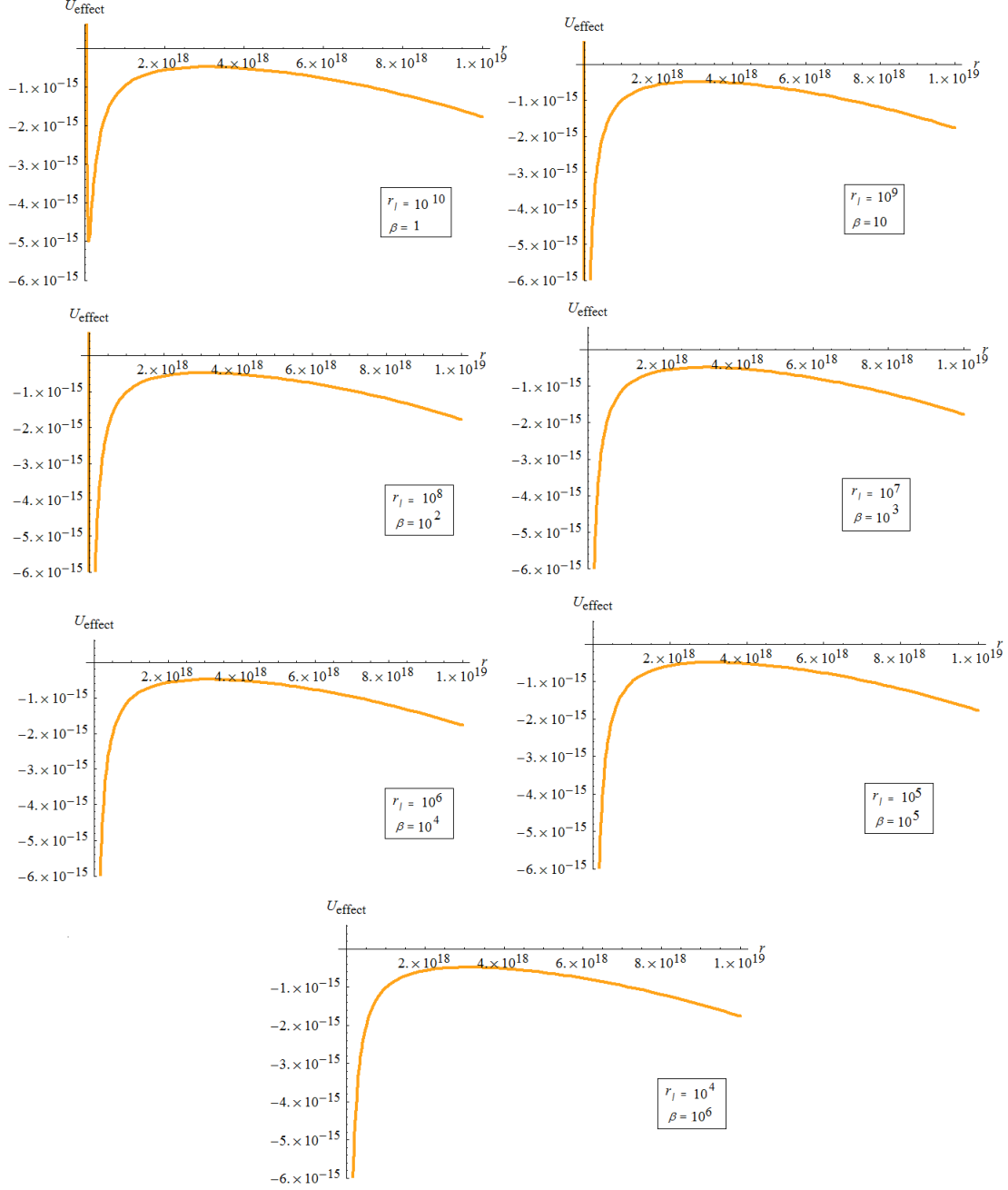


FIG. 3: Effective potential scales of the order $r \approx r_0$ and $L \gg r_s$.

$$z_1 \approx 4r_\Lambda L_{\text{max}} \left(1 - \frac{1}{4\beta^2} \right) \quad (135)$$

by taking the root square, then we get:

$$\sqrt{z_1} \approx 2\sqrt{r_\Lambda L_{\text{max}} \left(1 - \frac{1}{4\beta^2} \right)} \quad (136)$$

for $\beta^2 \gg 1$, we can expand in a series:

$$\sqrt{z_1} \approx 2\sqrt{r_\Lambda L_{max}} \left(1 - \frac{1}{8\beta^2}\right) \quad (137)$$

for z_2 and z_3 we can use analogous procedures:

$$z_2 \approx -2\frac{r_\Lambda L_{max}}{\beta} \sinh(Ln(8\beta^3)^{1/3}) - 2\sqrt{3}i\frac{r_\Lambda L_{max}}{\beta} \cosh(Ln(8\beta^3)^{1/3}) \quad (138)$$

or:

$$z_2 \approx -2r_\Lambda L_{max} \left(1 - \frac{1}{4\beta^2}\right) - 2\sqrt{3}ir_\Lambda L_{max} \left(1 + \frac{1}{4\beta^2}\right) \quad (139)$$

for z_3 , we have:

$$z_3 \approx -2r_\Lambda L_{max} \left(1 - \frac{1}{4\beta^2}\right) + 2\sqrt{3}ir_\Lambda L_{max} \left(1 + \frac{1}{4\beta^2}\right) \quad (140)$$

using the same reasoning done in order to obtain eq. 124 for the case i), as the cases must have compatibility with each other, then we can express directly the complex numbers given in eq. 139 and eq. 140 as:

$$z_2 \approx 2r_\Lambda L_{max} \left(-1 - \sqrt{3}i\right) + 2\frac{r_\Lambda L_{max}}{4\beta^2} \left(1 - \sqrt{3}i\right) \quad (141)$$

or:

$$z_2 \approx 4r_\Lambda L_{max} e^{-i2\pi/3} + \frac{r_\Lambda L_{max}}{\beta^2} e^{-i\pi/3} \quad (142)$$

and:

$$z_3 \approx 4r_\Lambda L_{max} e^{i2\pi/3} + \frac{r_\Lambda L_{max}}{\beta^2} e^{i\pi/3} \quad (143)$$

taking the root square of eq. 142 and eq. 143, we get:

$$\sqrt{z_2} \approx \sqrt{4r_\Lambda L_{max} e^{-i2\pi/3} + \frac{r_\Lambda L_{max}}{\beta^2} e^{-i\pi/3}} \quad (144)$$

and:

$$\sqrt{z_3} \approx \sqrt{4r_\Lambda L_{max} e^{i2\pi/3} + \frac{r_\Lambda L_{max}}{\beta^2} e^{i\pi/3}} \quad (145)$$

if we execute the expansion in a Taylor series, for large values of β^2 , we get:

$$\sqrt{z_2} \approx 2\sqrt{r_\Lambda L_{max}} e^{-i\pi/3} + \frac{1}{4\beta^2} \sqrt{r_\Lambda L_{max}} \quad (146)$$

and:

$$\sqrt{z_3} \approx 2\sqrt{r_\Lambda L_{max}} e^{i\pi/3} + \frac{1}{4\beta^2} \sqrt{r_\Lambda L_{max}} \quad (147)$$

replacing eqns. 137, 146 and 147 inside of 50, the only physically acceptable real solution is given by:

$$r_3^Z = \frac{3}{8} \frac{\sqrt{r_\Lambda L_{max}}}{\beta^2} \quad (148)$$

introducing eq. 56 inside of this result and using the definition $L = \frac{L_{max}}{\beta}$, we get:

$$r_3^Z = \frac{L^2}{r_s} \quad (149)$$

if we compare this result with the one obtained in eq. 90, it is clear that the effective potential goes to zero before the second critical value (circular orbit condition) which is completely consistent. Then at the value given by eq. 90, the effective potential must be negative. There is no other real zero for the effective potential in this second case and at big values of r , which means that at the third critical point given by eq. 92 (which is a maximum), the effective potential is still negative. If we evaluate the effective potential eq. 40 at the distance 90, term by term, we get:

$$\begin{aligned} U_{eff} \left(\frac{2L^2}{r_s} \right) = & - \left(\frac{2}{3} \right)^{2/3} \left(\frac{r_s}{r_\Lambda} \right)^{2/3} \beta^2 - \frac{1}{6} \left(\frac{3}{4} \right)^{4/3} \left(\frac{r_s}{2r_\Lambda} \right)^{2/3} \frac{1}{\beta^4} \\ & + \frac{1}{2} \left(\frac{2}{3} \right)^{2/3} \left(\frac{r_s}{r_\Lambda} \right)^{2/3} \beta^2 - 16 \left(\frac{1}{3} \right)^{8/3} \left(\frac{r_s}{r_\Lambda} \right)^{4/3} \beta^4 \end{aligned} \quad (150)$$

for most of the situations $\beta \gg 1$, then:

$$U_{eff} \left(\frac{2L^2}{r_s} \right) \approx - \frac{1}{2} \left(\frac{2}{3} \right)^{2/3} \left(\frac{r_s}{r_\Lambda} \right)^{2/3} \beta^2 - 16 \left(\frac{1}{3} \right)^{8/3} \left(\frac{r_s}{r_\Lambda} \right)^{4/3} \beta^4 < 0 \quad (151)$$

the last term in this equation can be neglected. If we want the two terms on the right-hand side of eq. 151 to be comparable it is necessary:

$$\frac{1}{2} \left(\frac{4}{3} \right)^{2/3} \left(\frac{r_s}{r_\Lambda} \right)^{2/3} \beta^2 \approx \left(\frac{4}{3} \right)^{8/3} \left(\frac{r_s}{r_\Lambda} \right)^{4/3} \beta^4 \quad (152)$$

solving for β , then:

$$\beta^2 \approx \left(\frac{9}{32} \right) \left(\frac{r_\Lambda}{r_s} \right)^{2/3} \quad (153)$$

for a source with a mass of the order of our sun:

$$\beta \approx 10^7 \quad (154)$$

which is outside of the possible range of values taken by β in order to keep the assumption $L \gg r_s$ as can be easily verified. Then equation 151 can be approximated as:

$$U_{eff} \left(\frac{2L^2}{r_s} \right) \approx - \frac{1}{2} \left(\frac{2}{3} \right)^{2/3} \left(\frac{r_s}{r_\Lambda} \right)^{2/3} \beta^2 < 0 \quad (155)$$

if we evaluate now the effective potential at the third critical point given in eq. 92, then we obtain term by term using the approximation 44:

$$U_{eff} \left(\left(\frac{3}{2} r_s r_\Lambda^2 \right)^{1/3} \right) \approx -\frac{1}{3^{1/3}} \left(\frac{r_s}{2r_\Lambda} \right)^{2/3} - \frac{9^{1/3}}{6} \left(\frac{r_s}{2r_\Lambda} \right)^{2/3} + \frac{1}{2\beta^2} \left(\frac{3}{4} \right)^{4/3} \frac{(r_s^2 r_\Lambda)^{2/3}}{(6r_s r_\Lambda^2)^{2/3}} \quad (156)$$

where we have used the definition $L = \frac{L_{max}}{\beta}$ and 56, for big values of β , we get approximately:

$$U_{eff} \left((3r_s r_\Lambda^2)^{1/3} \right) \approx - \left(\frac{r_s}{r_\Lambda} \right)^{2/3} < 0 \quad (157)$$

which is equal in order of magnitude to U_{eff} evaluated at the saddle point given in eq. 75 (see also eq. 130).

V. CONCLUSIONS

In this manuscript I could re derive some of the astrophysical scales (effects) due to the presence of a Cosmological Constant Λ . The scale $r_0 = \left(\frac{3}{2} r_s r_\Lambda^2 \right)^{1/3}$ was derived again by using a more rigorous method in order to get the first order corrections due to the angular momentum of a test particle moving around a source. By taking into account its first order correction, it takes the value $r_2^* = r_0' \approx \left(\frac{3}{2} r_s r_\Lambda^2 \right)^{1/3} - \frac{1}{4\beta^2} (3r_s r_\Lambda^2)^{1/3}$, where $\beta = \frac{L_{max}}{L}$. This is the new contribution of the manuscript. Additionally, I demonstrated that the condition for getting the saddle point position r_x is just equivalent to setting to zero the Discriminant $D = 0$ for the fourth order polynomial. From this limit condition, the maximum angular momentum L_{max} in order to get bound orbits is obtained. There is a saddle point position r_x for the effective potential when the test particle has an angular momentum L_{max} .

Appendix A: Minimum angular momentum with $\Lambda = 0$

In the effective potential given by eq. 29, we can find the first derivative, which is given by eq. 30, also, we can find from this the second derivative given by:

$$\frac{d^2 U_{eff}(r)}{dr^2} = -\frac{r_s}{r^3} + 3\frac{L^2}{r^4} - 6\frac{r_s L^2}{r^5} = 0 \quad (\text{A1})$$

where we have sent to zero the second derivative of the effective potential; this equation can be rewritten as:

$$r^2 - 3\left(\frac{L^2}{r_s}\right)r + 6L^2 = 0 \quad (\text{A2})$$

the equality between this equation and that gotten in 31, for which we got in that moment the condition for circular orbits, implies:

$$r^2 - \frac{2L^2}{r_s}r + 3L^2 = r^2 - \frac{3}{2}\left(\frac{2L^2}{r_s}\right)r + 6L^2 \quad (\text{A3})$$

solving for r , we get:

$$r = 3r_s \quad (\text{A4})$$

replacing this result inside eq. A2, we get:

$$9r_s^2 - 9L^2 + 6L^2 = 0 \quad (\text{A5})$$

solving for L , we get:

$$L_{min} = \sqrt{3}r_s \quad (\text{A6})$$

which is the minimum angular momentum.

Appendix B: General solution of a fourth-order polynomial

The standard form of a fourth-order equation can be taken to be:

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0 \quad (\text{B1})$$

to get the reduced form of this equation, we need to do the following variable change:

$$y \equiv x + \frac{B}{4A} \quad (\text{B2})$$

solving for x , we get:

$$x = y - \frac{B}{4A} \quad (\text{B3})$$

replacing this result inside of B1, we get:

$$A\left(y - \frac{B}{4A}\right)^4 + B\left(y - \frac{B}{4A}\right)^3 + C\left(y - \frac{B}{4A}\right)^2 + D\left(y - \frac{B}{4A}\right) + E = 0 \quad (\text{B4})$$

developing parenthesis, we get:

$$\begin{aligned}
& A \left(y^4 - \left(\frac{B}{A} \right) y^3 + \frac{3}{8} \left(\frac{B}{A} \right)^2 y^2 - \frac{1}{16} \left(\frac{B}{A} \right)^3 y + \left(\frac{B}{4A} \right)^4 \right) + \\
& B \left(y^3 - \frac{3}{4} \left(\frac{B}{A} \right) y^2 + \frac{3}{16} \left(\frac{B}{A} \right)^2 y - \frac{1}{64} \left(\frac{B}{A} \right)^3 \right) + C \left(y^2 - \frac{B}{2A} y + \frac{1}{16} \left(\frac{B}{A} \right)^2 \right) \\
& + D \left(y - \frac{B}{4A} \right) + E = 0 \quad (\text{B5})
\end{aligned}$$

regrouping common factors and dividing by A, we get:

$$y^4 + \left(-\frac{3}{8} \left(\frac{B}{A} \right)^2 + \frac{C}{A} \right) y^2 + \left(\frac{1}{8} \left(\frac{B}{A} \right)^3 - \frac{CB}{2A^2} + \frac{D}{A} \right) y - \frac{3}{256} \left(\frac{B}{A} \right)^4 + \frac{1}{16} \frac{C}{A^3} B^2 - \frac{DB}{4A^2} + \frac{E}{A} = 0 \quad (\text{B6})$$

the standard reduced fourth-order equation is given by:

$$y^4 + Py^2 + Qy + K = 0 \quad (\text{B7})$$

comparing this form with B6, we get:

$$\begin{aligned}
P &= \frac{C}{A} - \frac{3}{8} \left(\frac{B}{A} \right)^2 \\
Q &= \frac{1}{8} \left(\frac{B}{A} \right)^3 - \frac{CB}{2A^2} + \frac{D}{A} \\
K &= \frac{E}{A} - \frac{DB}{4A^2} + \frac{1}{16} \frac{CB^2}{A^3} - \frac{3}{256} \left(\frac{B}{A} \right)^4
\end{aligned} \quad (\text{B8})$$

the associated cubic equation of B7 is:

$$z^3 + 2Pz^2 + (P^2 - 4K)z - Q^2 = 0 \quad (\text{B9})$$

where P, Q and K are given in B8. Given the solutions of the associated third-order equation B9, the solutions for the reduced equation B7 are given by:

$$\begin{aligned}
y_1 &= \frac{1}{2}(\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}) \\
y_2 &= \frac{1}{2}(\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}) \\
y_3 &= \frac{1}{2}(-\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}) \\
y_4 &= \frac{1}{2}(-\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3})
\end{aligned} \quad (\text{B10})$$

also, the solutions of the associated third-order equation B9, must satisfy the following:

$$Q = \sqrt{z_1}\sqrt{z_2}\sqrt{z_3} \quad (\text{B11})$$

Appendix C: General solution of a third-order polynomy

The standard form of a third-order polynomy is given by:

$$ax^3 + bx^2 + cx + d = 0 \quad (\text{C1})$$

the normal form is gotten dividing by a this result as follows:

$$x^3 + rx^2 + sx + t = 0 \quad (\text{C2})$$

where obviously, we have:

$$r = \frac{b}{a} \quad s = \frac{c}{a} \quad t = \frac{d}{a} \quad (\text{C3})$$

with the extra condition $a \neq 0$. The reduced form of the third-order equation C2, requires the change of variable:

$$y \equiv x + \frac{r}{3} \quad (\text{C4})$$

and the reduced form is given by:

$$y^3 + py + q = 0 \quad (\text{C5})$$

with the corresponding coefficients given by:

$$p = s - \frac{r^2}{3} \quad q = \frac{2}{27}r^3 - \frac{rs}{3} + t \quad (\text{C6})$$

where r , s and t are given in C3. On the other hand, it is necessary to establish a classification criteria for the reduced third-order equation given in C5, the criteria is based in a parameter D given by:

$$D \equiv \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \quad (\text{C7})$$

also for the same classification, it is necessary to find an auxiliary parameter given by:

$$R \equiv \text{sign}(q)\sqrt{\frac{|p|}{3}} \quad (\text{C8})$$

where q and p are defined in C5 and C6. On the other hand, we define an auxiliary angle ϕ , which is defined depending of the following cases:

Case i). $p < 0, D \leq 0$.

In this case, the auxiliary angle is defined as:

$$\cos\phi \equiv \frac{q}{2R^3} \quad (\text{C9})$$

with the corresponding solutions:

$$\begin{aligned} y_1 &= -2R\cos\frac{\phi}{3} \\ y_2 &= -2R\cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right) \\ y_3 &= -2R\cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right) \end{aligned} \quad (\text{C10})$$

Case ii). $p < 0, D > 0$.

In this case, the auxiliary angle is defined as:

$$\cosh\phi \equiv \frac{q}{2R^3} \quad (\text{C11})$$

and the corresponding solutions are:

$$\begin{aligned} y_1 &= -2R\cosh\frac{\phi}{3} \\ y_2 &= R\cosh\frac{\phi}{3} + i\sqrt{3}R\sinh\frac{\phi}{3} \\ y_3 &= y_2^* = R\cosh\frac{\phi}{3} - i\sqrt{3}R\sinh\frac{\phi}{3} \end{aligned} \quad (\text{C12})$$

Case iii). $p > 0, D > 0$.

In this section, we define the auxiliary angle to be:

$$\sinh\phi \equiv \frac{q}{2R^3} \quad (\text{C13})$$

with the corresponding solutions:

$$\begin{aligned} y_1 &= -2R\sinh\frac{\phi}{3} \\ y_2 &= R\sinh\frac{\phi}{3} + i\sqrt{3}R\cosh\frac{\phi}{3} \\ y_3 &= y_2^* = R\sinh\frac{\phi}{3} - i\sqrt{3}R\cosh\frac{\phi}{3} \end{aligned} \quad (\text{C14})$$

In the mentioned three cases, D is defined in equation C7; p is defined in equations C5 and C6, and the parameter R is defined in equation C8.

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